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A Homomorphism Theorem for the Poincaré Group†

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Abstract

Jordan demonstrated that the group of homogeneous transformations of degree one in $\mathbb{R}^5$ is homomorphic to the symmetry group of the Einstein-Maxwell equations in vacuum. It is shown that the Jordan homomorphism theorem is also applicable to the inhomogeneous general linear group. Consequently, the Poincaré group is homomorphic to the group of homogeneous transformations of degree one in a five-dimensional space.

Introduction

Jordan (1945), in his projective theory of relativity, showed that the symmetry group of the Einstein-Maxwell equations for the combined electromagnetic and gravitational fields in vacuum is isomorphic to a subgroup of the group of homogeneous transformations of degree one in a five-dimensional space. This is the basis of the unified field theory of gravitation and electromagnetism based on a projective formalism, the so-called projective theory of relativity (Jordan, 1955).

It turns out (Evans & Sen, 1973) that the symmetry group of the Einstein-Maxwell equations has a semi-direct product structure—it is a semi-direct product of the group of coordinate transformations in space-time and the group of gauge transformations. And, it can be shown that the Jordan homomorphism theorem is also applicable to other groups with a semi-direct product structure. Thus the inhomogeneous general linear group $IGL(n, \mathbb{R})$ (and consequently the Poincaré group) is isomorphic to a subgroup of the group $H(n + 1, \mathbb{R})$ of homogeneous transformations of degree one in $\mathbb{R}^{n+1}$.

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The Homomorphism Theorem

Consider the Einstein–Maxwell equations† for the combined electromagnetic and gravitational fields in vacuum

$$\left.\begin{array}{l} G_{ik} + \kappa E_{ik} = 0 \\ f^{ik}_{;k} = 0, \qquad f_{ik} = \phi_{i,k} - \phi_{k,i} \end{array}\right\} \tag{1}$$

where ϕ_i is the electromagnetic four-potential, $E_{ik} = f_{il}f_k^{\ l} - \frac{1}{4}g_{ik}f_{lm}f^{lm}$, the energy-momentum tensor of the electromagnetic field f_{ik} and G_{ik}, the Einstein tensor for space-time with metric g_{ik}. Equation (1) is, of course, invariant under the pseudogroup K of C^∞ transformations in $\mathbb{R}^4$

$$K \ni \Lambda : x^k \to x^{k'} = x^{k'}(x^k) \tag{2}$$

Strictly speaking, K is a pseudogroup, because coordinate transformations in a C^∞ manifold, i.e. C^∞ diffeomorphisms between open sets in $\mathbb{R}^n$ do not quite satisfy all the properties of a group. For a precise definition of pseudogroup of transformations see Kobayashi & Nomizu (1963).

But (1) is also invariant under the (Abelian) group E of gauge transformations

$$E \ni [\phi] : \phi_i \to \phi_i + \phi_{,i} \tag{3}$$

where ϕ is a scalar function in $\mathbb{R}^4$. The symmetry group of the Einstein–Maxwell equations is therefore the combined pseudogroup G, a typical element of which will be denoted by $g = (\Lambda, [\phi])$ to mean a gauge transformation $[\phi]$ *followed* by a coordinate transformation Λ, i.e.

$$(\Lambda, [\phi]) : \phi_i(x^k) \xrightarrow{[\phi]} \phi_i(x^k) + \phi(x^k)_{,i} \xrightarrow{\Lambda} (\phi_i(x^k) + \phi(x^k)_{,i})x^i_{,i'} \tag{4}$$

Note that the unit element of E is $[c]$, where c is any constant. Denote by e the unit element of K. If we consider a gauge transformation $[\phi_1]$ followed by a coordinate transformation Λ_1 and then $[\phi_2]$ and Λ_2—all in that order, then the product rule for G is easily seen to be

$$(\Lambda_2, [\phi_2]) \cdot (\Lambda_1, [\phi_1]) = (\Lambda_2\Lambda_1, [\phi_2 \circ \Lambda_1 + \phi_1]) \tag{5}$$

where $\phi_2 \circ \Lambda_1$ is given by:

$$x \xrightarrow{\Lambda_1} x' \xrightarrow{\phi_2} \phi_2(x'(x))$$

The inverse rule is therefore

$$(\Lambda, [\phi])^{-1} = (\Lambda^{-1}, [-\phi \circ \Lambda^{-1}]) \ldots \tag{6}$$

† Here comas and semicolons denote partial and covariant derivatives respectively, and the summation convention is used throughout unless specified otherwise. Latin indices take values $i, k = 1, 2, 3, 4$ and Greek indices $\mu, \lambda = 0, 1, 2, 3, 4$.

We see that $G = KE$, i.e. $(\Lambda, [\phi])$ can be written uniquely as $(\Lambda, [\phi]) = (\Lambda, [c]) \cdot (e, [\phi])$.

K induces an automorphism of E as follows:

$$\begin{aligned}(\Lambda, [c]) : (e, [\phi]) &\to (\Lambda, [c])(e, [\phi])(\Lambda, [c])^{-1} \\ &= (\Lambda, [\phi])(\Lambda^{-1}, [-c \circ \Lambda^{-1}]) \\ &= (e, [\phi \circ \Lambda^{-1}]) \in E\end{aligned}$$

And E is a normal subpseudogroup of G, because

$$\begin{aligned}(\Lambda, [\phi])^{-1}(e, [\psi])(\Lambda, [\phi]) &= (\Lambda^{-1}, [-\phi \circ \Lambda^{-1}])(e, [\psi])(\Lambda, [\phi]) \\ &= (\Lambda^{-1}, [-\phi \circ \Lambda^{-1}])(\Lambda, [\psi \circ \Lambda + \phi]) \\ &= (e, [\psi \circ \Lambda]) \in E\end{aligned}$$

We thus have the following structure theorem for G.

Theorem 1. *G is a semi-direct product of K and E.*

Let now $H(5, \mathbb{R})$ be the (pseudo) group of homogeneous transformations of degree one in $\mathbb{R}^5$. That is, an element $h \in H(5, \mathbb{R})$ is of the form

$$h : \chi^\mu \to \chi^{\mu'} = \chi^{\mu'}(\chi^\mu), \qquad \mu = 0, 1, 2, 3, 4 \tag{7}$$

where $\chi^{\mu'}(\chi^\mu)$ are invertible, *homogeneous functions of degree one. h* can be also written as

$$h : \chi^\mu \to \chi^{\mu'} = \chi^\mu f^{(\mu)}(\chi^\mu) \qquad \text{(no summation)} \tag{8}$$

where $f^{(\mu)}(\chi^\mu)$ are invertible, homogeneous functions of degree zero. $H(5, \mathbb{R})$ has the following subgroups:

$$J = \left\{ h \in H(5, \mathbb{R}) | h : \chi^\mu \to \chi^{\mu'} = \chi^\mu f(\chi^\mu) = \chi^\mu F\left(\frac{\chi^1}{\chi^0}, \dots \frac{\chi^4}{\chi^0}\right) \right\}$$

$$N = \left\{ h \in H(5, \mathbb{R}) | h : \begin{array}{l} \chi^0 \to \chi^{0'} = \chi^0 \\ \chi^k \to \chi^{k'} = \chi^0 f^k \left(\frac{\chi^1}{\chi^0}, \dots \frac{\chi^4}{\chi^0}\right) \end{array} \right\} \tag{9}$$

J is a normal subgroup of N. There exist the following homomorphisms

$$\begin{array}{lll} J \to E, & J \ni j \to [\phi] = [\log F] & \\ N \simeq K, & N \ni n \quad \Lambda : x^k \to x^{k'} = f^k(x^k) & \end{array} \tag{10}$$

and thus proves the Jordan homomorphism theorem.

Jordan Homomorphism Theorem. $H(5, \mathbb{R})$ is homomorphic to the symmetry group G of the Einstein–Maxwell equations in vacuum.

Now $H(5, \mathbb{R})$ is a semi-direct product of J and N. This suggests that the Jordan Homomorphism Theorem is also applicable to other groups with a semi-direct product structure. We can easily prove the following theorem.

Theorem 2. The inhomogeneous general linear group $IGL(n, \mathbb{R})$ is homomorphic to $H(n + 1, \mathbb{R})$.

Proof. $IGL(n, \mathbb{R})$ is a semi-direct product of $GL(n, \mathbb{R})$ and Υ^n where

$$\left.\begin{aligned} GL(n, \mathbb{R}) \ni \Lambda &: x^i \to a_{ik}x^k \\ \Upsilon^n \ni \mathbf{A} &: x^i \to x^i + A^i \end{aligned}\right\} i, k = 1, 2, \ldots, n \tag{11}$$

Consider now the following subgroup of $H(n + 1, \mathbb{R})$

$$\left.\begin{aligned} \chi^0 &\to \chi^0 \\ \chi^i &\to \chi^0 f^i\left(\frac{\chi^1}{\chi^0}, \ldots \frac{\chi^n}{\chi^0}\right) = \chi^0 a_{ik}\left(\frac{\chi^k}{\chi^0}\right) \end{aligned}\right\} \tag{12}$$

which is a subgroup of the corresponding (for general n) subgroup N of (9) and is isomorphic to $GL(n, \mathbb{R})$.

Let now $\mathbf{A} \in \Upsilon^n$ and $\phi_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\mathbf{x} \to \phi_{\mathbf{A}}(\mathbf{x})$ is linear in $\mathbf{A}$, i.e.

$$\phi_{\mathbf{A}_1 + \mathbf{A}_2}(\mathbf{x}) = \phi_{\mathbf{A}_1}(\mathbf{x}) + \phi_{\mathbf{A}_2}(\mathbf{x}) \ldots \tag{13}$$

For example, take $\phi_{\mathbf{A}}(\mathbf{x}) = (\mathbf{A} \,.\, \mathbf{x})$.

Consider now the mapping $\Upsilon^n \to H(n + 1, \mathbb{R})$ given by

$$\left.\begin{aligned} \mathbf{A} &\to \exp\left(\phi_{\mathbf{A}}(\mathbf{x})\right) = F(\mathbf{x}) \\ \chi^\mu &\to \chi^\mu F\left(\frac{\chi^1}{\chi^0}, \ldots \frac{\chi^n}{\chi^0}\right) \end{aligned}\right\} \ldots \tag{14}$$

Thus Υ^n is isomorphic to a subgroup of the corresponding (for general n) subgroup J of (9).

Instead of taking a linear function $\phi_{\mathbf{A}}$ one can also consider a function $\psi_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}$ such that $\psi_{\mathbf{A}_1 + \mathbf{A}_2}(\mathbf{x}) = \psi_{\mathbf{A}_1}(\mathbf{x}) \,.\, \psi_{\mathbf{A}_2}(\mathbf{x})$ and consider the mapping $\Upsilon^n \to (n + 1, \mathbb{R})$ given by

$$\left.\begin{aligned} \mathbf{A} &\to \psi_{\mathbf{A}}(\mathbf{x}) = F(\mathbf{x}) \\ \chi^\mu &\to \chi^\mu F\left(\frac{\chi^1}{\chi^0}, \ldots, \frac{\chi^n}{\chi^0}\right) \end{aligned}\right\} \tag{15}$$

For example, take $\psi_{\mathbf{A}}(\mathbf{x}) = \exp(\mathbf{A} \,.\, \mathbf{x})$. Q.E.D.

The following corollary is then obvious.

Corollary. The Poincaré group in n-dimensions is homomorphic to $H(n + 1, \mathbb{R})$.

Conclusion

It is thus possible to embed the Poincaré group non-trivially in a larger group of transformations of a five-dimensional space. It has been suggested (*Review of Modern Physics*, 1965) that embedding of curved space-times in a higher dimensional pseudoeuclidean space may provide an understanding of the informal symmetry of elementary particles. It is possible that considerations of non-trivial embedding of the Poincaré group may also provide a similar understanding.

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